

People often ask me what $i2\pi$ means. To me, $i2\pi$ represents the pleasure of stringing together simple principles to arrive at a beautiful understanding of the nature of something. Here I am going to share with you the way I think about $i2\pi$ in a mathematical context. This doesn't represent the full history or complete derivation, but it is how I like to think about things. If you want to learn more, Wikipedia is a good place to start. If you want to see things through my eyes, for just a moment, stick around. If you scroll ahead it may look a little scary to the mathematically uninitiated, but we won't be using anything more complicated than mid-range high school math to get there. And I think the journey is worthwhile.

To understand what $i2\pi$ is all about we really have to start with Euler's constant, e . Euler's constant is a special number in mathematics and it appears in many equations across mathematics, physics, statistics and economics because it has a number of unique properties. One of my favourite properties is the relationship between e^x and its derivative $\frac{de^x}{dx}$, namely

$$\frac{de^x}{dx} = e^x.$$

Those who have taken high-school calculus take this for granted. Those who didn't or who have forgotten it are probably scratching their heads. For their sake, and my own, let's quickly review derivatives.

Derivatives are a convenient way of describe the slope of a line. Take the equation for the line $y = 2x$, then for each unit increase in x , we get 2 units of increase in y . The slope of a line is the ratio between the change in the output of the function that describes it with respect to the change in the input. In this case, x is the input and y is the output, and increasing x by 1 increases y by 2, so the slope is 2:1 or simply 2. The notation $\frac{df(x)}{dx}$ describes the slope of some function $f(x)$ as x changes. The equation $y = 2x$ has the same slope for all values of x , so we say the slope is a constant. More complicated functions, like $f(x) = x^2$ (in the figure below) are curved, so the slope changes as we change x . Without going too deep into calculus, it is known that $\frac{dx^2}{dx} = 2x$. In fact, for any k

$$\frac{dx^k}{dx} = kx^{k-1}.$$

For example, $\frac{dx^7}{dx} = 7x^6$.

When you take the derivative (find the slope) of most functions, the answer is usually some modified form of the function you started with. However, the exponential function, e^x is the simplest example of the case where the derivative is equal to the function. Up to this point, we haven't even worked out what the numerical value of e is, but let us try to define e by starting with the fact that $\frac{de^x}{dx} = e^x$. To do this, we will need to take a detour into the world of factorials.

Imagine that we have 5 books that we want to place on a shelf. How many different ways can we arrange them? Working methodically, from left to right, there are 5 possible books we could put in the leftmost spot on the shelf. Once we choose the first book to place there, we are left with 4 possible books to place along side it. And once we choose that book, 3 books remain. After placing the third book, only 2 more remain, and so on. This means there must be $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways of arranging those 5 books. If we had 100 books, it would take up too much paper to write down $100 \times 99 \times 98 \times \dots \times 1$, so we use the shorthand $100!$, which we pronounce '100 factorial'. One hundred factorial is a big number. If we built a book arranging robot that could do one billion arrangements per second, and had one billion of them running in parallel since the big bang, they still wouldn't be finished trying all the possible ways to arrange the books. Thank god for the Dewey Decimal system, eh?

Ok. So factorials are just a shorthand way of writing down a special type of successive multiplication. We can use the formula for the slope of x^k to find the slope of

$$f(x) = \frac{x^k}{k!}.$$

We can re-arrange this to be

$$f(x) = \frac{x^k}{k!} = \frac{1}{k!}x^k = Cx^k.$$

If you multiply some function by a constant, C , then the slope is also multiplied by C . From the rule we saw 2 paragraphs ago, we now that

$$\frac{df(x)}{dx} = \frac{d}{dx}Cx^k = Ckx^{k-1}$$

We also know that $k! = k \times (k-1)!$. For example, $6! = 6 \times 5! = 6 \times 5 \times 4 \times \dots \times 1$. So

$$\frac{d}{dx} \frac{x^k}{k!} = \frac{kx^{k-1}}{k(k-1)!}$$

Cancel out the k in the numerator and denominator, and we get

$$\frac{df(x)}{dx} = \frac{x^{k-1}}{(k-1)!}.$$

Now let's tackle a slightly more complicated function,

$$g(x) = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

The first 2 terms of this function are pretty simple. Recall that factorials count the number of ways of arranging objects. There is only one way to arrange zero books, so $0! = 1$. Also recall from math class that $x^0 = 1$, so

$$\frac{x^0}{0!} = 1.$$

There is only one way of arranging one book and $x^1 = x$, so

$$\frac{x^1}{1!} = x.$$

Therefore

$$g(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

I know this comes out of nowhere, but just go with flow. What is the derivative of this function? Derivatives are additive so we can just do each bit individually and add them together. The derivative of 1 is 0, as it is a flat line - hence no slope. The derivative of x is 1, as it is a line with a 45 degree slope. And we just worked out the rest in the previous paragraph, so

$$\frac{dg(x)}{dx} = 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Hey, wait up. If we drop the leading 0, which we can, then that's just $g(x)$ again.

$$\frac{dg(x)}{dx} = 0 + g(x) = g(x).$$

To see this requires some majorly deep and majorly simple insight. When things are deep AND simple, they are beautiful. What just happened was that even though the first term of the sum disappeared by turning into zero, the rest of the sum remained. Because we defined $g(x)$ to go on from $\frac{x^3}{3!} + \dots$ to infinity, for each term that drops off the front, there are an infinite number of terms to make up for it. Infinity less one is still infinity.

Recall that we defined e^x to have the property that

$$\frac{de^x}{dx} = e^x$$

and we now have a function where

$$\frac{dg(x)}{dx} = g(x).$$

These properties are the same. The function has itself as its derivative. And it just so happens that $g(x)$ is e^x . To understand why, we need to look at Taylor's Theorem. So point your browser to Wikipedia if you are so inclined and join us in the next paragraph to continue.

Great. Welcome back. The point of this post is to explain $i2\pi$ and so far we have only covered e , so let's get a move on and have a look at i and π . I assume everyone is cool with 2.

What is i ? i is the square root of negative one.

$$\sqrt{-1} = i$$

So, $i^2 = -1$. When I first encountered i I asked a family friend / math professor to explain it to me. All the books I had read just talked about 'complex numbers' and I wanted to understand what made them 'complex.' She explained to me that they aren't complex, in the sense that complex means difficult. They are just different to the normal numbers we usually encounter. In school you would have learned that the square root of negative numbers is undefined. But they turn up so frequently that man invented a new class of numbers to allow us to define them. Numbers are just symbols for the abstract concept of quantity. And i is just a symbol for the square root of negative one. While it is not possible to have i books or bananas, we can still do mathematics with i and end up with real world numbers. For example,

$$i^2 = i \times i = \sqrt{-1} \times \sqrt{-1} = -1.$$

And $i^4 = i^2 \times i^2 = -1 \times -1 = 1$. So while we can't buy i bananas, we can buy i^4 bananas, because i^4 is 1. As you keep on raising i to higher and higher powers, a pattern emerges. $i^1 = i$, $i^2 = -1$, $i^3 = i^2 \times i = -i$ and $i^4 = 1$. When we look at i^5 we find $i^5 = i^4 \times i = 1 \times i = i$, and the pattern repeats. For no apparant reason, lets sum up all the powers of i :

$$\begin{aligned} & i + i^2 + i^3 + i^4 + i^5 + i^6 \dots \\ &= i + (-1) + (-1 \times i) + (-1 \times -1) + (-1 \times -1 \times i) + (-1 \times -1 \times -1) \dots \\ &= i - 1 - i + 1 + i - 1 \dots \end{aligned}$$

To see the pattern more clearly, lets split up the odd an even terms

$$\begin{aligned} & i^2 + i^4 + i^6 + i^8 + \dots = -1 + 1 - 1 + 1 - \dots \\ & i^1 + i^3 + i^5 + i^9 + \dots = i - i + i - i + \dots \end{aligned}$$

Out of sheer curiosity, lets find out what pattern would we get if we expanded e^{ix} using the formula we found before.

$$e^{ix} = 1 + ix + \frac{i^2 x^2}{2!} + \frac{i^3 x^3}{3!} + \frac{i^4 x^4}{4!} + \dots$$

We know that the pattern in the i 's comes out nicely if we split out the evens and odds, so lets call the even part of the right hand side of the equation $a(ix)$ and $b(ix)$ for the odds:

$$a(ix) = 1 + \frac{i^2 x^2}{2!} + \frac{i^4 x^4}{4!} + \frac{i^6 x^6}{6!} + \dots$$

Now using the $-1 + 1 - 1 + 1 - \dots$ pattern, we get

$$a(ix) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Likewise for the odd terms,

$$b(ix) = i \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right)$$

Now when you went across to Wikipedia to check out Taylor's Theorem, you will have seen that

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Which is totally the same as our $a(ix)$. I may not be hip and fresh with the Jonas brothers and skateboarding, but I know cool when I see it, and that is cool. We also know that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Which means that $b(ix) = i \times \sin(x)$. Putting a and b together, we find that

$$e^{ix} = a(ix) + b(ix) = \cos(x) + i \times \sin(x)$$

Now that we have gone from summing polynomials to trigonometry, it may be coming clear where the π fits in. π is a special number that defines the ratio between a circle's diameter and its circumference. If a circle has a diameter of one furlong, then its circumference will be π furlongs. π is also used to measure angles, the same way as degrees are. In a circle there are 360 degrees, but mathematicians like to say that a circle has 2π radians. That is, if you have a circle with a radius of 1 foot, then the circumference will be 2π feet. If you were to walk around this circle through 1/8ths of its circumference, you will have moved 45 degrees, or $\frac{\pi}{4}$ radians.

The sine and cosine take an angle in radians and tell you the x and y coordiants of that point on a circle. If you walk 45 degrees anti-clockwise around a circle starting from the point $(1,0)$, then you will end up at position

$$\left(\cos\left(\frac{\pi}{8}\right), \sin\left(\frac{\pi}{8}\right)\right).$$

If you walk 2π radians around the circle, you will have gone 360 degrees and end up where you started. So $(\cos(2\pi), \sin(2\pi)) = (1, 0)$

So the function $e^{i\omega}$ tells us the coordinates of where one ends up after walking ω radians around a circle of radius 1. We have been writing down our coordiates as (x, y) , where $x = \cos\omega$ and $y = \sin(\omega)$, but as we found out earlier,

$$e^{i\omega} = \cos(\omega) + i\sin(\omega) = x + iy.$$

If we think of i as being a symbol to represent our distance in the y direction, then we can convert from $x + iy$ to (x, y) . And if we walk around a full circle and end up at the beginning point of $(1, 0)$, we can convert back to $1 + i \times 0 = 1$. Therefore

$$e^{i2\pi} = 1.$$

I like $i2\pi$ as this term comes up across a range of mathematical equations and you can go a long way in learning about mathematics form understanding its origins.